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ON CERTAIN CONSTRUCTIONS OF LIE (SUPER) ALGEBRAS AND SYMMETRIC (SUPER) SPACES ASSOCIATED WITH (ε, δ)-FREUDENTHAL-KANTOR TRIPLE SYSTEMS

-In homage of Professor Susumu Okubo (1930-2015)

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Abstract

The main purpose of this paper is to study constructions of Lie algebras and symmetric spaces associated with (ε, δ)-Freudenthal-Kantor triple systems based on certain concept of ternary product (triple product) without using root systems in Lie algebras. Next, by means of the constructions, we will give several examples of bisymmetric (super)spaces.

Keywords: triple systems, symmetric spaces, Lie (super)algebras construction.
MSC classification: Primary 17A40 ternary compositions, 17B60 Lie (super)algebras associated with other structures, 17C50 Jordan structures associated with other structures. Secondary 53B25 Local submanifolds, 53C35 Differential geometry of symmetric spaces.

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Introduction and Motivation

This note is to deal with certain constructions of Lie (super)algebras and of symmetric spaces based on our subjects called a triple system (or ternary algebra). We give several examples of triple systems (or ternary products) equipped with the balanced property, then by using this triple product's concept, we exhibit a construction of symmetric spaces (Lie triple systems) and simple Lie algebras without using root systems, in particular, our purpose is to consider a construction of bisymmetric spaces from balanced Freudenthal-Kantor triple systems. That is, it seems that this study in nonassociative algebra field with respect to Jordan and Lie structures is in close contact with a symmetric (super)space equipped with complex structure of differencial geometry, since the tangent space of the symmetric (super)space is a δ-Lie triple system ($\delta = \pm 1$).
Hereby we exhibit main our results described in Section 2 and Section 4 of this article.

Theorem A *For a balanced (1,1)-Freudenthal-Kanmtot triple system U, the Lie algebra $L(U)$ and the Lie triple system $T(U)$ associated with U, we have subspaces of quaternionic symmetric spaces as follows;*

$$F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1).$$

Theorem B *For a balanced (-1,-1)-Freudenthal-Kantor triple system U, the Lie superalgebra $L(U)$and the anti-Lie triple system $T(U)$ associated with U, we have subspaces of symmetric superspaces as follows;*

- $osp(2|2)/(A_1 \oplus gl(1)) < osp(1|4)/(A_1 \oplus A_1) < F(4)/(A_1 \oplus B_3)$,
- $osp(1|2)/A_1 < osp(3|2)/(A_1 \oplus A_1) < G(3)/(A_1 \oplus G_2)$.

From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [12], [51], [64], [66], [69]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many

mathematical and physical subjects ([6], [11]-[14], [16], [23], [27], [29], [30], [33], [35], [36], [37], [50], [59], [60], [65], [70]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [7]-[9], [21], [24], [25], [45]-[50], [55]-[58], [61]-[63]) by using the standard embedding method ([23], [53], [54], [62], [68]). In particular, the generalized Jordan triple system of second order, or $(-1,1)$-Freudenthal Kantor triple system (for short $(-1,1)$-FKTS), is a useful concept ([14]-[22], [46]-[49], [52], [67]) for the constructions of simple Lie algebras, while the $(-1,-1)$-FKTS plays the same role ([7], [23], [26], [28], [31], [34], [35]) for the construction of Lie superalgebras, while the δ-Jordan Lie triple systems act similarly for that of Jordan superalgebras ([24], [25], [61]). Specially, we have constructed a model of basic Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ ([23], [26], [28]).

For a simple(primary) example of our motivations, if we decompose the associative binary product of the matrix algebra $Mat(n,n;\Phi)$ in its symmetric and skew-symmetric parts

$$xy = \frac{1}{2}(xy+yx) + \frac{1}{2}(xy-yx),$$

then the second term leads to the Lie algebra $gl(n,\Phi)$ with the product $[x,y] = xy - yx$, also, by the triple product $[xyz] = [[x,y],z]$, it leads to the Lie triple system, and the first term leads to the Jordan algebra $Mat(n,n;\Phi)$ with the product $x \cdot y = \frac{1}{2}(xy+yx)$, also by the triple product $\{xyz\} = \frac{1}{2}(x^t yz + z^t yx)$, where ${}^t x$ denote the transpose matrix of x, it leads to the Jordan triple system.

As a final comment of this introduction, we provide well-known results due to O. Loos as follows; if A is a unital commutative Jordan algebra with a unital element e, that is, satisfying $(xy)x^2 = x(y(x^2))$, and $xy = yx$, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system with $\{xey\} = xy$, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), and $\{xyz\} = \{zyx\}$ (this relation is

called a commutative identity, since $xy = \{xey\} = \{yex\} = yx$) and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system.

Briefly summarizing this article, we will generalize these results and exhibit examples of Lie (super)algebras associated with Freudenthal-Kantor triple systems which is a generalization of the Jordan triple system. Toward to its applications, in particular, we will give a construction of symmmetric (super)spaces with an almost complex structure (i.e., equipped with Nijenhuis operator). And we will provide an idea of quaternionic bisymmetric spases associated with our constructions.

Roughly describing, we have an illustration for our concept ;

Algebraic structures $\Longleftrightarrow$ Geometric structures.

For examples, it seems that there are certain algebraic structures associated with symmetric, R-symmetric, homogeneous spaces, totally geodesic manifolds, and symmetric domains, further Yang-Baxter equations etc. Hence, the notations of triple systems appeared in this article is utility with respect to the characterization of geometry or mathematical physics. In particular, it emphasizes that our objects are to give a representation of symmetric (super)spaces from concepts of triple systems (ternary algebras).

1 Definitions and Preliminaries

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order) ([46]-[50]) .

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{2}$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \ i.e., \ K(a,c)b = 0. \tag{3}$$

The JTS is a special case in the GJTS of 2nd order since $K(x,y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [14], [15], [19], [23], [29], [38] [68] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{5}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{6}$$

is called an $(\varepsilon,\delta)-Freudenthal-Kantor\ triple\ system$ (for short (ε,δ)-FKTS). An (ε,δ)-FKTS is said to be *unitary* if $Id \in \{K(a,b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized* FKTS (for short GFKTS), while the identity (5) is called the *second order condition* (this condition needs to construct of 5-graded Lie (super)algebras).

Remark From the relation Eq. (6), we note that

$$K(b,a) = -\delta K(a,b). \tag{7}$$

A triple system is called a (α,β,γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha < x,y > z + \beta < y,z > x + \gamma < z,x > y,$$

where $< x,y >$ is a bilinear form such that $< x,y >= \kappa < y,x >$, $\kappa = \pm 1$, $\alpha,\beta,\gamma \in \Phi$.

With respect to a classification of this triple system, we would like to refer ([42]).

An (ε,δ)-FKTS is said to be *balanced* if there is a bilinear form $< x,y > \in \Phi^*$ such that $K(x,y) =< x,y > Id$, that is, $dim_\Phi \{K(x,y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x,y) =< x,y > Id$) is unitary, since $Id \in \{K(x,y)\}_{span}$.

Triple products are denoted by (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([7], [14]-[20], [22], [23], [26], [28], [48]).

For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc], a,b,c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{gathered} [abc] = -\delta[bac], \\ [abc] + [bca] + [cab] = 0, \\ [ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]], \end{gathered} \tag{8}$$

where $a,b,x,y,z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by ([15]). Note that the set $L(V,V)$ of all left multiplications $L(x,y)$ of V is a Lie subalgebra of $Der\ V$, where we denote by $L(x,y)z = [xyz]$.

Theorem 1.1 ([14]-[17], [23]) *Let $(U(\varepsilon,\delta), < xyz >)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz >=< JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] :=< xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Remark Note that for the case of $\varepsilon = -1, \delta = 1$ and $K(x,y) = 0$, we have a special case in Prop.1.1, that is, it implies that $J = Id$, $\{xyz\}$ is the JTS and $[xyz] = \{xyz\} - \{yxz\}$ is the LTS described in Introduction.

Corollary ([14]) *Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\binom{a}{b}\binom{c}{d}\binom{e}{f}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\binom{e}{f}. \tag{10}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie

superalgebra (if $\delta = -1$), $L(U(\varepsilon,\delta)) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$;

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x,y \in U(\varepsilon,\delta) \right\}_{span}.$$

We note that $L(X,Y)Z = [XYZ] = [[X,Y],Z],\ \forall\ X,Y,Z \in T(\varepsilon,\delta)$.

From $K(x,y) \in End\ U(\varepsilon,\delta)$, we use the following notation:

$$\mathbf{k} := \{K(x,y) | x,y \in U(\varepsilon,\delta)\}, \{EFG\} := EFG + GFE,\ \forall\ E,F,G \in \mathbf{k}.$$

Then, we may make a structure of a JTS $\mathbf{k}$ with respect to the triple product $\{EFG\} \in \mathbf{k}$, hence $[EFG] = \{EFG\} - \{FEG\}$ has a structure of LTS ([21]).

We next introduce an analogue of Nijenhuis tentor in differential geometry defined by

$$N(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] + J^2[X,Y], \forall X,Y \in T(\varepsilon,\delta)$$

and $J = \begin{pmatrix} 0 & \varepsilon \\ -\delta & 0 \end{pmatrix}$, that is $J^2 = -\varepsilon\delta Id$, hence if $J^2 = -Id$, then this (the case of $\varepsilon\delta = 1$) has a structure of allmost complex on the δ LTS $T(\varepsilon,\delta)$.

Theorem 1.2 *Let $U = U(\varepsilon,\delta)$ be a (ε,δ)-FKTS, $T(U) = T(U(\varepsilon,\delta))$ be the δ-LTS and $L(U) = L(U(\varepsilon,\delta))$ be the standard embedding Lie (super)algebra associated with U. Then the following are equivalent:*

(i) $N(X,Y) = 0, \forall X,Y \in T(\varepsilon,\delta)$,

(ii) $\varepsilon\delta L(y,x) - \varepsilon L(x,y) = K(x,y),\ \forall x,y \in U(\varepsilon,\delta)$.

This $J \in End\ T(\varepsilon,\delta)$ may generalize on $\tilde{J} \in End\ L(U)$ defined by

$$\tilde{J} := JD(X,Y)J^{-1} \oplus JZ,\ \ \forall X,Y,Z \in T(\varepsilon,\delta).$$

Then we note that $\tilde{J}$ has an interesting property, for example, an automorphism of $L(U)$ associated with U.

Theorem 1.3 *For a (ε,δ)-FKTS U and $L(U)$ as in above Proposition, assuming $\varepsilon = \delta$ and $K(x,y) = L(y,x) - \varepsilon L(x,y)$, then the elements $f =$*

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl(2)$ *(i.e.,* $[f,g] = h$, $[f,h] = -2f$, $[g,h] = 2g$ *) are derivations of* $L(U)$. *Furtheremore, the Lie (super)algebra* $L(U)$ *has a structure of* $sl(2)$*-module.*

Remark We note that $L(U) = L(U(\varepsilon,\delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon,\delta) \oplus U(\varepsilon,\delta) = L_{-1} \oplus L_1 = T(\varepsilon,\delta)$ (δ-LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon,\delta), T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon,\delta)$) equipped with $[L_i, L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems without using root systems (for a Lie superalgebra, refer to ([10], [13], [65])). Also from another viewpoint, this construction can be represented by the concept of a normal triality algebra (see [34], [35]).

This section is a survey of our papers with respect to the triple systems and the construction of Lie (super)algebras mainly, if it need, the readers would like to see the earlier our references of therein.

Remark Note that if $K(x,y) = < x,y > Id$ and $\delta = 1$, from Eq.(5), then we obtain $K(x,y) = L(y,x) - \varepsilon L(x,y)$, and by $\delta = 1$, we have $K(x,y) = -K(y,x)$. That is, balanced and $\delta = 1 \Longrightarrow \varepsilon = 1$. Also, balanced and $\delta = -1 \Longrightarrow \varepsilon = -1$. Furtheremore, if $< x,y >$ is nondegenerate, the standard embedding Lie algebra $L(U)$ is simple ([16], [19], [20], [32]).

Following ([17], [19], [20], [36], [37]), with respect a generalized vector matrix $X = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in M(V)$, where $\alpha, \beta \in \Phi$, $\mathbf{a}$, $\mathbf{b} \in V = H_3(\mathbf{O})$, where $\mathbf{O}$ is an octonion algebra over the field Φ, in order to make as self-contained as possible, we recall first the definition of an outer product $\times$ and a bilinea form $B(x,y)$ in the exceptional Jordan algebra $H_3(\mathbf{O})$ as follows;

$$A \times B := 2A \cdot B - (Tr\ A)B - (Tr\ B)A + [(Tr\ A)(Tr\ B) - Tr\ (A \cdot B)]E$$

and $B(A,B) := Tr\ (A \cdot B)$ for $A, B \in H_3(\mathbf{O})$, where $A \cdot B = \frac{1}{2}(AB + BA)$. Secondary we consider either the set of matrices $Mat(m,n;\Phi)$ with coefficients in Φ or the set of the generalized vector matrix;

$$M(V) = \{ \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} | \alpha, \beta \in \Phi, \mathbf{a}, \mathbf{b} \in V = Mat(m,n;\Phi)\ or\ H_3(\mathbf{O}) \}.$$

In this $M(V)$, we introduce a bilinear operator $\circ$

$$\begin{pmatrix} \alpha_1 & \mathbf{a_1} \\ \mathbf{b_1} & \beta_1 \end{pmatrix} \circ \begin{pmatrix} \alpha_2 & \mathbf{a_2} \\ \mathbf{b_2} & \beta 2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha_1\alpha_2 + B(\mathbf{a}_1, \mathbf{b}_2) & \alpha_1\mathbf{a}_2 + \beta_2\mathbf{a}_1 + \mathbf{b}_1 \times \mathbf{b}_2 \\ \alpha_2\mathbf{b}_1 + \beta_1\mathbf{b}_2 + \mathbf{a}_1 \times \mathbf{a}_2 & \beta_2\beta_1 + B(\mathbf{a}_2, \mathbf{b}_1) \end{pmatrix}. \qquad (11)$$

Next we define the involutions

$$P: \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} -\alpha & \mathbf{a} \\ -\mathbf{b} & \beta \end{pmatrix} \text{ and } - : \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta & \mathbf{a} \\ \mathbf{b} & \alpha \end{pmatrix}.$$

Then by using above notations, we can obtain a triple product on $M(V)$ as follows:

$$(\spadesuit) \quad < x_1x_2x_3 >= x_1 \circ (\overline{Px_2} \circ x_3) + x_3 \circ (\overline{Px_2} \circ x_1) - Px_2 \circ (\overline{x_1} \circ x_3)$$

where $x_i = \begin{pmatrix} \alpha_i & \mathbf{a_i} \\ \mathbf{b_i} & \beta_i \end{pmatrix}$ $(i = 1, 2, 3)$.

By straightfoward calculations, for the case of $H_3(\mathbf{O})$, we have the following theorem, but long proof, we omit it (see, the details [17], [19]).

Theorem 1.4 *Under the notations as in above, the vector space* $(M(V), < xyz >)$ *is a balanced Freudenthal-Kantor triple system with* $\varepsilon = 1$, $\delta = 1$.

For balanced Freudenthal-Kantor triple systems, we have the property of Propositions 1.1, 1.2 and 1.3, because balanced is a special case of (ε, δ)-Freudenthal-Kantor triple system, that is, either the case of $\varepsilon = 1$, $\delta = 1$ or $\varepsilon = -1$, $\delta = -1$.

From now on, we assume that Φ is an algebraically closed field of characteristic zero, unless otherwise specified. Thus we can construct the simple exceptional Lie algebras, the Lie triple systems as follows and bisymmetric spaces are described by next section.

$X = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ (balanced case, that is, $dim\ M(V) = 56$, $dim\ L_{-2} = dim\ L_2 = dim\ \{K(x,y)|x,y \in M(V)\}_{span} = 1$), where $\mathbf{a}$, $\mathbf{b} \in H_3(\mathbf{O})$ (exceptional Jordan algebra with 27 dimension) and α, $\beta \in \Phi$. The Lie algebra constructed from this triple system $U = M(V)$ is the following; $L(U) \cong E_8$, $L_{-2} \oplus L_0 \oplus L_2 \cong E_7 \oplus A_1$, $L_0 \cong E_7 \oplus gl(1)$, $U = M(V) = L_{-1}$.

To change the notation $H_3(\mathbf{O}) \to H_3(\mathfrak{A})(=B)$, here $\mathfrak{A}$ is Hurwitz algebras over Φ, let $^\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} \Phi & B \\ B & \Phi \end{pmatrix} = A$, with respect to the $dim\ B$, and Lie algebras $L(A)$ obtained from B, we have the following.

$dim B$	1	6	9	15	27
$dim A$	4	14	20	32	56
$dim L(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

where $dim\ \mathfrak{A} = 8, 4, 2$ and 1 mean that algebras $\mathfrak{A}$ are the octonion algebra $\mathbf{O}$ over the field Φ, a quaternion $\mathbf{H}$ over Φ, a separable quadratic field over Φ, and the field Φ, respectively. Also, $dim\ B = 1$ means that B is the field Φ.

For this case's E_8, considering the extended Dynkin diagram, we have $L_{-1} \oplus L_1 \cong L(U)/(L_{-2} \oplus L_0 \oplus L_2) = E_8/(A_1 \oplus E_7)$ with dim $L_{-1} \oplus L_1$=112;

◎–□–○–○–○–○–○–○, □ omitted $\cong A_1 \oplus E_7 = Der\ (L_{-1} \oplus L_1)$.
  | ◎ is the highest root
  ○

Remark This construction of balanced type with $dim\ M(V) = 56$ has been first studied by H. Freudenthal (called a metasymmetric geometry equipped with notations of $P \times Q$ and $\{P, Q\}$). That is, this concept is characterized from the triple system (or a ternary algebra) and this $U = M(V)$ is called a generalized Zorn's vector matrix ([13]-[16],[18],[35] and the references of therein).

Remark ([16]) Note that for simple exceptional balanced Freudenthal-Kantor triple systems $U = M(V)$ ($\varepsilon = 1,\ \delta = 1$), the Lie triple systems $T(U)$ and the standard embedding Lie algebras $L(U)$, if $dim\ U = n$, we have the following;

$dim\ Der\ U = 3n(n+1)/(n+16)$,

$dim\ Anti - Der\ U = dim\ \{L(x,y) - L(y,x) | x, y \in U\}_{span} = 1$,

$dim\ T(U) = 2n,\ and\ dim\ L(U) = (5n^2 + 38n + 48)/(n+16)$.

Here $L(U)$ are $G_2,\ F_4,\ E_6,\ E_7,\ E_8$ types. For example, if $n = 56$, then we obtain $dim\ Der\ U = 133$ and $dim\ L(U) = 248$.

2 Bisymmetric spaces associated with exceptional simple Lie algebras

Following the results due to O. Loos ([51]) or W. Bertram ([5]) with respect to symmetric spaces, it is known that any symmetric space G/H may be identified with some Lie triple system T (as the tangent space of the symmetric space is a Lie triple system).

We consider a concept of bisymmetric space $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ on Lie triple subsystems pair defined as follows:

From the balanced type of exceptional cases, we may define a concept of the pair $(T_\alpha, T_\beta, T_\gamma, T_\delta)$.

$$(\heartsuit) \quad (dim\ T_\delta + 16)/dim\ T_\gamma = (dim\ T_\gamma + 16)/dim\ T_\beta =$$

$$(dim\ T_\beta + 16)/dim\ T_\alpha = 2,\ and\ T_\alpha < T_\beta < T_\gamma < T_\delta$$

It is said to be a *quaternionic bisymmetric space.* This means that

$$(\heartsuit\heartsuit) \quad F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1)$$

and the dimension of bisymmetric spaces of balanced type;

$$28, \quad 40, \quad 64, \quad 112 \qquad respectively.$$

With respect to extended Dynkin diagrams (symmetric spaces) of E_7, E_6, F_4; $\circ\!-\!\circ\!-\circ-\circ-\circ-\boxed{\circ}-\circledcirc$, $\boxed{\circ}$ omitted $\cong D_6 \oplus A_1$, and $\circledcirc$ is the highest root.

$\circ-\circ-\circ-\circ-\circ-\circ$, $\boxed{\circ}$ omitted $\cong A_5 \oplus A_1$, and $\circledcirc$ is the highest root.

$\circledcirc-\boxed{\circ}-\circ \Rightarrow \circ-\circ$, $\boxed{\circ}$ omitted $\cong C_3 \oplus A_1$, and $\circledcirc$ is the highest root.

Here $A_1, A_5, B_4, C_3, D_5, D_6$ mean classical simple Lie algebras.

From now on, we call a quaternionic symmetric space for the balanced case, in the sence of W. Bertram ([6]).

In final results of this section, we deal with the construction of Lie algebra of G_2 type and the bisymmetric space associated with the balanced Freudenthal-Kantor triple system.

Let

$$M(V) = \{\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} | \alpha, \beta, \gamma, \delta \in V = \Phi\},$$

$B(\alpha, \beta) = \alpha\beta$ and the cross product $\times$ be identically zero. Then by Theorem 1.4, we can obtain the structure of the balanced triple system on $M(V)$. If $< x, y >$ is nondegenerate, then from the dimensionality of the standard embedding Lie algebra $L(M(\Phi))$ with $L_{-1} = M(V)$, it follows that the Lie algebra is G_2 type.

Indeed, we have $dim_\Phi < L(x,y) >_{span} = 4$, and $K(x,y) = L(y,x) - L(x,y)$, so $dim_\Phi < K(x,y) >_{span} = 1$, since $K(x_1, x_2) =< x_1, x_2 >= \beta_1\alpha_2 - \alpha_1\beta_2 + \gamma_1\delta_2 - \delta_1\gamma_2$.

Next, in above $M(V)$, let $\gamma = \delta = 0$, then by Eq. (♠) in section 1, we have

$$< x_1x_2x_3 >= \begin{pmatrix} 2\alpha_1\beta_2\alpha_3 + \alpha_2\beta_1\alpha_3 & 0 \\ 0 & -(2\alpha_2\beta_1\beta_3 + \alpha_1\beta_2\beta_3) \end{pmatrix},$$

$$\text{for } x_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}, \ (i = 1, 2, 3).$$

From $K(x_1, x_2) =< x_1, x_2 >= \beta_1\alpha_2 - \alpha_1\beta_2$, we have the fact that the standard embedding Lie algebra $L(M(\Phi))$ is A_2 type.

Hence, we can obtain a bisymmetric space such that $dim\ T_\delta = 2dim\ T_\gamma$;

$$T_\gamma = A_2/(A_1 \oplus gl(1)) \ < T_\delta = G_2/(A_1 \oplus A_1).$$

The extended Dynkin diagram of $Der\ T_\delta$ is the following.

$\circledcirc - \boxed{\circ} \equiv > \circ$, $\boxed{\circ}$ omitted $\cong A_1 \oplus A_1$. and $\circledcirc$ is highest root.

3 Bisymmetric spaces associated with classical simple Lie algebras

In this section, we consider bisymmetric spaces associated with balanced Freu-denthal-Kantor triple systems of classical type.

(♯) A_n type (the dimension is $n(n+2)$):

Let $M_A(V)$ be the set

$$M_A(V) := \{\begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{b} & 0 \end{pmatrix} | \mathbf{a},\ \mathbf{b} \in V = Mat(1, n-1)\}, \begin{pmatrix} 0 & \mathbf{a} \\ {}^t\mathbf{b} & 0 \end{pmatrix} \in Mat(n, n; \Phi).$$

We may introduce a triple product as well as exceptional cases as follows:

$$(\spadesuit\spadesuit) \quad < x_1x_2x_3 >= x_1 \circ (\overline{Px_2} \circ x_3) + x_3 \circ (\overline{Px_2} \circ x_1) - Px_2 \circ (\overline{x_1} \circ x_3)$$

where $x_1 = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_2} & 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_2} & 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_2} & 0 \end{pmatrix} \in M_A(V)$.

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_2} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_2} & 0 \end{pmatrix}$

$= \begin{pmatrix} B(\mathbf{a_1}, \mathbf{b_2}) & 0 \\ 0 & B(\mathbf{a_2}, \mathbf{b_1}) \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_2} & 0 \end{pmatrix} \circ \begin{pmatrix} B(\mathbf{a_1}, \mathbf{b_2}) & 0 \\ 0 & B(\mathbf{a_2}, \mathbf{b_1}) \end{pmatrix} =$

$\begin{pmatrix} 0 & B(\mathbf{a_2}, \mathbf{b_1})\mathbf{c_1} \\ B(\mathbf{a_1}, \mathbf{b_2})\mathbf{c_2} & 0 \end{pmatrix} \in M_A(V)$, since $B(\mathbf{a_2}, \mathbf{b_1})\mathbf{c_1}$,$B(\mathbf{a_1}, \mathbf{b_2})\mathbf{c_2} \in Mat(1, n-1; \Phi)$. And we denote $B(\mathbf{a_1}, \mathbf{b_2}) = \mathbf{a_1}{}^t\mathbf{b_2} = \mathbf{b_2}{}^t\mathbf{a_1} \in Mat(1, 1; \Phi) = \Phi$, also, ${}^t\mathbf{a}$ denotes the transepose matrix of $\mathbf{a}$ and the cross product $\times$ is identically zero, $\overline{x_1} = x_1$, since the diagonal elements of the vector matrix $M_A(V)$ are 0, the notation P is same in section one. By straightfoward calculations, we have

$$(L(x_1, x_2)+L(x_2, x_1))x_3 = \begin{pmatrix} 0 & -\mathbf{c_1}{}^t\mathbf{d} \\ \mathbf{c_2}\mathbf{d} & 0 \end{pmatrix}, (L(x_1, x_2)-L(x_2, x_1))x_3 = \begin{pmatrix} 0 & -\alpha\mathbf{c_1} \\ \alpha\mathbf{c_2} & 0 \end{pmatrix},$$

where $\mathbf{d} = 2{}^t\mathbf{b_1}\mathbf{a_2} + 2{}^t\mathbf{a_1}\mathbf{b_2} + (\mathbf{a_1}{}^t\mathbf{b_2} + \mathbf{b_1}{}^t\mathbf{a_2})I_{n-1,n-1} \in Mat(n-1, n-1; \Phi)$ and $\alpha = \mathbf{a_1}{}^t\mathbf{b_2} - \mathbf{b_1}{}^t\mathbf{a_2} \in \Phi$.

Theorem 3.1 *The triple system $M_A(V)(= U)$ defined by $(\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = 1$, $\delta = 1$.*

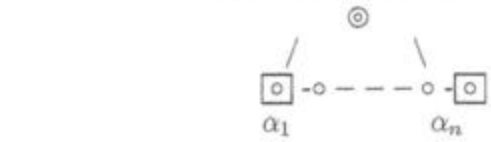
Proof For $\varepsilon = 1,\ \delta = 1$, it is enough to verify the relations (4) and (5) for the case of balanced property ($K(x,y) =< x,y > Id$). These relations are straightforward calculations, but we omit it, as it is long. □

Therefor, we get $< L(x,y) >_{span} \cong A_{n-2} \oplus gl(1) \oplus gl(1)$ (as a Lie algebra), and $K(x_1,x_2) = -L(x_1,x_2) + L(x_2,x_1) = -\alpha = -(\mathbf{a_1}^t\mathbf{b_2} - \mathbf{b_1}^t\mathbf{a_2}) \in \Phi$. Hence, the standard embedding Lie algebra implies that $L(U) \cong A_n$ type's Lie algebra associated with $U = M_A(V)$, because $dim_\Phi\ L(U) = dim_\Phi\ < L(x,y) >_{span} + 2\ dim_\Phi\ < K(x,y) >_{span} + 2\ dim_\Phi\ M_A(V) = (n-1)^2 + 1 + 2 + 4(n-1) = n(n+2)$. From this A_n, we may construct a bisymmetric space's series ($T_\alpha,\ T_\beta,\ T_\gamma,\ T_\delta$), that is, there is a Lie triple subsystem's series as follows:

If $n = 2m-1,\ m = 2k-1,\ k = 2l-1$, then we have a bisymmetric space's series;

$T_\alpha = A_l/(A_{l-2} \oplus A_1 \oplus gl(1)) < T_\beta = A_k/(A_{k-2} \oplus A_1 \oplus gl(1)) < T_\gamma = A_m/(A_{m-2} \oplus A_1 \oplus gl(1)) < T_\delta = A_l/(A_{n-2} \oplus A_1 \oplus gl(1)) <$ and $dim\ T_\delta = 2dim\ T_\gamma = 4dim\ T_\beta = 8dim\ T_\alpha$, where $dim\ T_\delta = 4(n-1),\ dim\ T_\gamma = 4(m-1),\ dim\ T_\beta = 4(k-1),\ and\ dim\ T_\alpha = 4(l-1)$.

Based on the Dynkin diagram of Lie algebra A_n type, the extended Dynkin diagram of corresponding $Der\ T_\delta$ is the following.



(##) B_n *and* D_n types (the dimension is $n(2n+1)$ and $n(2n-1)$, respectively):

$$M_{B,D}(V) := \{\begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} | \mathbf{a} \in V = Mat(2,p)\}, \begin{pmatrix} 0 & \mathbf{a} \\ {}^t\mathbf{a} & 0 \end{pmatrix} \subset Mat(p+2, p+2; \Phi).$$

We may introduce a triple product as well as exceptional cases as follows:

$$(\spadesuit\spadesuit) \quad < x_1x_2x_3 >= x_1 \circ (\overline{Px_2} \circ x_3) + x_3 \circ (\overline{Px_2} \circ x_1) - Px_2 \circ (\overline{x_1} \circ x_3)$$

where $x_1 = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_1} & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_1} & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_1} & 0 \end{pmatrix} \in M_{B,D}(V)$.

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_1} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_1} & 0 \end{pmatrix} = \begin{pmatrix} {}^t(\sigma_0 B(\mathbf{a_1}, \mathbf{b_1})) & 0 \\ 0 & B(\mathbf{b_1}, \mathbf{a_1})\sigma_0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_1} & 0 \end{pmatrix} \circ \begin{pmatrix} B(\mathbf{a_1}, \mathbf{b_1}) & 0 \\ 0 & B(\mathbf{b_1}, \mathbf{a_1}) \end{pmatrix}$
$= \begin{pmatrix} 0 & B(\mathbf{b_1}, \mathbf{a_1})\mathbf{c_1} \\ B(\mathbf{a_1}, \mathbf{b_1})\mathbf{c_1} & 0 \end{pmatrix} \in M(V)$, since $B(\mathbf{a_1}, \mathbf{b_1})\mathbf{c_1} \in Mat(2, p; \Phi)$, and $B(\mathbf{b_1}, \mathbf{a_1})\mathbf{c_1} \in Mat(2, p; \Phi)$. We denote the notations by $B(\mathbf{a_1}, \mathbf{b_1}) = \mathbf{a_1}{}^t\mathbf{b_1}$, $B(\mathbf{b_1}, \mathbf{a_1}) = \mathbf{b_1}{}^t\mathbf{a_1} \in Mat(2, 2; \Phi)$, also, ${}^t\mathbf{a}$ denotes the transepose matrix of $\mathbf{a}$ and the cross product $\times$ is identically zero, $\overline{x_1} = x_1$, $\sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By straightforward calculations, we have the following.

Theorem 3.2 *Then the triple system $M_{B,D}(V)$ defined by (♠♠) is a balanced Freudenthal-Kantor triple system with $\varepsilon = 1, \delta = 1$.*

Furthermore, we have $L(x_1, x_2)x_3 = \begin{pmatrix} 0 & \mathbf{c_1 d} \\ \mathbf{c_1 d} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{e c_1} \\ \mathbf{e c_1} & 0 \end{pmatrix}$, where $\mathbf{d} = -({}^t\mathbf{b_1}\sigma_0\mathbf{a_1} + {}^t\mathbf{a_1}\sigma_0\mathbf{b_1}) \in Mat(p, p; \Phi)$, $\mathbf{e} = -\mathbf{a_1}{}^t\mathbf{b_1}\sigma_0 \in Mat(2, 2; \Phi)$. Hence, $dim_\Phi < L(x, y) >_{span} = p(p-1)/2 + 4$, $dim_\Phi < K(x, y) >_{span} = 1$ Therefore, if $p = 2n - 3$, then $< L(x, y) >_{span} \cong B_{n-2} \oplus A_1 \oplus gl(1)$ (as Lie algebra), $< K(x, y) >_{span} \cong gl(1) = \Phi$ and the standard embedding Lie algebra $L(M_{B,D}(V))$ is the B_n type. If $p = 2n - 4$, we get $< L(x, y) >_{span} \cong D_{n-2} \oplus A_1 \oplus gl(1)$ (as a Lie algebra), and $K(x_1, x_2) = -L(x_1, x_2) + L(x_2, x_1) =< x, y > \in \Phi$. Hence, the standard embedding Lie algebra $L(M_{B,D}(V))$ is the D_n type. From these B_n and D_n types, we may construct a bisymmetric space's series (T_α, T_β, T_γ, T_δ), that is, Lie triple subsystems as follows:

If $n = 3(m-1)$, $m = 3(k-1)$, $k = 3(l-1)$ (the case of B_n), then we have bisymmetric spaces satisfying

$T_\alpha = B_l/(B_{l-2} \oplus A_1 \oplus A_1) < T_\beta = B_k/(B_{k-2} \oplus A_1 \oplus A_1) < T_\gamma = B_m/(B_{m-2} \oplus A_1 \oplus A_1) < T_\delta = B_n/(B_{n-2} \oplus A_1 \oplus A_1)$ and $dim\, T_\delta = 3dim\, T_\gamma = 9dim\, T_\beta = 27dim\, T_\alpha$, where $dim\, T_\delta = 4(2n-3)$, $dim\, T_\gamma = 4(2m-3)$, $dim\, T_\beta = 4(2k-3)$, *and* $dim\, T_\alpha = 4(2l-3)$.

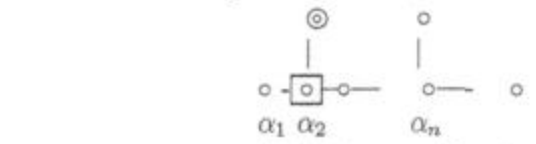
The extended Dynkin diagram of corresponding $Der\, T_\delta$ is the following.

◎ is the highest root omitted □ $\cong B_{n-2} \oplus A_1 \oplus A_1 = Der\, T_\delta$

α_1 α_n

If $n = 2(m-1)$, $m = 2(k-1)$, $k = 2(l-1)$ (the case of D_n), then we have a quaternionic bisymmetric spaces satisfying

$T_\alpha = D_l/(D_{l-2} \oplus A_1 \oplus A_1) < T_\beta = D_k/(D_{k-2} \oplus A_1 \oplus A_1) < T_\gamma = D_m/(D_{m-2} \oplus A_1 \oplus A_1) < T_\delta = D_n/(D_{n-2} \oplus A_1 \oplus A_1) <$ and $dim\ T_\delta = 2dim\ T_\gamma = 4dim\ T_\beta = 8dim\ T_\alpha$. The extended Dynkin diagram of corresponding $Der\ T_\delta$ is the following.



(###) C_n type (the dimension is $n(2n+1)$):

$M_C(V) := \{\begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} | \mathbf{a} \in V = Mat(1, 2(n-1))\}, \begin{pmatrix} 0 & \mathbf{a} \\ {}^t\mathbf{a} & 0 \end{pmatrix} \in Mat(2n-1, 2n-1; \Phi)$. We may introduce a triple product as well as exceptional cases as follows:

(♠♠) $< x_1x_2x_3 >= x_1 \circ (\overline{Px_2} \circ x_3) + x_3 \circ (\overline{Px_2} \circ x_1) - Px_2 \circ (\overline{x_1} \circ x_3)$

where $x_1 = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_1} & 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_1} & 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_1} & 0 \end{pmatrix} \in M_C(V)$.

Here the product $\circ$ is defined by $x \circ y = \begin{pmatrix} 0 & \mathbf{a_1} \\ \mathbf{a_1} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{b_1} \\ \mathbf{b_1} & 0 \end{pmatrix} = \begin{pmatrix} B(\mathbf{a_1}, \mathbf{b_1})) & 0 \\ 0 & B(\mathbf{b_1}, \mathbf{a_1}) \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{c_1} \\ \mathbf{c_1} & 0 \end{pmatrix} \circ \begin{pmatrix} B(\mathbf{a_1}, \mathbf{b_1}) & 0 \\ 0 & B(\mathbf{b_1}, \mathbf{a_1}) \end{pmatrix} = \begin{pmatrix} 0 & B(\mathbf{b_1}, \mathbf{a_1})\mathbf{c_1} \\ B(\mathbf{a_1}, \mathbf{b_1})\mathbf{c_1} & 0 \end{pmatrix} \in M(V)$, since $B(\mathbf{a_1}, \mathbf{b_1})\mathbf{c_1}$ and $B(\mathbf{b_1}, \mathbf{a_1})\mathbf{c_1} \in Mat(1, 2(n-1); \Phi)$. And we denote $B(\mathbf{a_1}, \mathbf{b_1}) = \frac{1}{2} < \mathbf{a_1}, \mathbf{b_1} >$, $B(\mathbf{b_1}, \mathbf{a_1}) = -B(\mathbf{a_1}, \mathbf{b_1}) \in Mat(1, 1; \Phi) = \Phi$, also, the cross product $\times$ is identically zero, $\overline{x_1} = x_1$. By straightforward calculations, we have the following.

Theorem 3.3 *The triple system $M_C(V)$ defined by (♠♠) is a balanced Freu-*
denthal-Kantor triple system with $\varepsilon = 1$, $\delta = 1$.

Furthermore, we have $L(x_1, x_2)x_3 = \begin{pmatrix} 0 & \mathbf{d} \\ \mathbf{d} & 0 \end{pmatrix}$, where $\mathbf{d} = -B(\mathbf{c_1}, \mathbf{b_1})\mathbf{a_1} - B(\mathbf{a_1}, \mathbf{b_1})\mathbf{c_1} - B(\mathbf{c_1}, \mathbf{a_1})\mathbf{b_1} \in Mat(1, 1; \Phi) = \Phi$. That is, we get

$< L(x,y) >_{span} \cong C_{n-1} \oplus gl(1)$ (as Lie algebra), and $K(x,y) = -L(x,y) + L(y,x) =< x,y > Id = gl(1) = \Phi$. Hence, $dim_\Phi \ < L(x,y) >_{span} = dim_\Phi \ < K(x,y) >_{span} = 1$, therefor, the standard embedding Lie algebra $L(M_C(V))$ is the C_n type.

From this C_n, we may construct a bisymmetric space's series $(T_\alpha,\ T_\beta,\ T_\gamma,\ T_\delta)$, that is, Lie triple subsystems as follows:

If $n = 2(m-1),\ m = 2(k-1),\ k = 2(l-1)$ (the case of B_n), then we have a quaternionic bisymmetric spaces satisfying

$T_\alpha = C_l/(C_{l-1} \oplus A_1) < T_\beta = C_k/(C_{k-1} \oplus A_1) < T_\gamma = C_m/(C_{m-1} \oplus A_1) < T_\delta = C_n/(C_{n-1} \oplus A_1)$ and $dim\ T_\delta = 2dim\ T_\gamma = 4dim\ T_\beta = 8dim\ T_\alpha$, where $dim\ T_\delta = 4(n-1),\ dim\ T_\gamma = 4(m-1),\ dim\ T_\beta = 4(k-1),\ and\ dim\ T_\alpha = 4(l-1)$.

The extended Dynkin diagram of corresponding $Der\ T_\delta$ is the following.
◎ => $\boxed{\circ}$ -○ – –○ <= ○ omitted $\boxed{\circ} \cong C_{n-1} \oplus A_1 = Der\ T_\delta$,◎ is highest root.

α_1 α_n

For the Lie triple systems pair $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ created by these methods, we note to call a *" quaternionic bisymmetric space"* associated with the balanced triple systems constructed in section 2 and 3.

Remark The concept of bisymmetric spaces can be continued without stopping for classical types if $n \gg 0$.

In final comments of this section, with respect to the construction of quaternionic symmetric spaces and simple Lie algebras eqipped with the $sl(2)$-module obtained from $< f,g,h >_{span} \cong A_1$, we emphasize that the construction created from balanced Freudenthal-Kantor triple systems is an important concept without using root systems. On the other hand, in ([42], [43], [44] and the references of therein), our subjects are considered by other triple systems equipped with no balanced types or by normal triality algebras (that is, by means of nonassociative algebras, in particular, by generalized vector matrix without using balanced cases).

In next sections, we will study about certain bisymmetric superspaces obtained from Lie superalgebras with associated with $(\varepsilon, -1)$ Freudenthal-Kantor triple systems.

4 Bisymmetric superspaces associated with balanced (-1,-1)-Freudenthal-Kantor triple systems

In this section, we consider bisymmetric superspaces associated with balanced Freudenthal-Kantor triple systems equipped with $\varepsilon = -1$ and $\delta = -1$ as well as section 2 and section 3 (the case of $\varepsilon = 1, \delta = 1$). The case of $T(-1,-1)$ associated with a $(-1,-1)$-Freudenthal-Kantor triple system is an anti-Lie triple system and so this means a symmetric superspace. For example, following concept of section 1 and section 2, it is known that there is a correspondence the symmetric space and the Lie triple system, because the tangent space of a symmetric space is a Lie triple system. Hence, we can generalize this concept to a correspondence of the symmetric superspace and the anti-Lie triple system.

Hereby for Lie superalgebras, we use notations as follows;
$A(m-1,n-1) = sl(m|n)$, $B(m,n) = osp(2m+1|2n)$, $C(n+1) = osp(2|2n)$, $D(m,n) = osp(2m|2n)$, in particular, $B(0,n) = osp(1|2n)$.

(♯♯♯♯) $B(n,1)$ types (the dimension is $(2n^2+5n+5)$) and $D(n,1)$ (the dimension is $(2n^2+3n+1)$) :

Let V be a vector space of the matrix set of $Mat(1,l;\Phi)$ and $< x,y >= x^t y \in \Phi^*$, $\forall x,y \in V$. Then we may define a triple product as follows;

$$(\spadesuit\spadesuit\spadesuit) \quad < xyz >= \frac{1}{2}(< x,y > z - < y,z > x + < z,x > y), \ \forall x,y,z \in V.$$

Theorem 4.1 *The triple system V defined by $(\spadesuit\spadesuit\spadesuit)$ is a balanced Freudenthal-Kantor triple system with $\varepsilon = -1$ and $\delta = -1$.*

Indeed, by straightforward calculations, we have the relatins $L(x,y) + L(y,x) = K(x,y) =< x,y > Id$ and Eq.(4), that is, a $(-1,-1)$-Freudenthal-Kantor triple system equipped with $dim_\Phi \ < K(x,y) >_{span} = 1$.

$\cdot\, l = 2n+1 \rightarrow$ *the standard embedding Lie superalgebra $L(V)$ is $B(n,1)$ type.*

$\cdot\, l = 2n \rightarrow$ *the standard embedding Lie superalgebra $L(V)$ is $D(n,1)$ type.*

For these extended Dynkin diagrams of $L(V)$, we have

$\circledcirc \Rightarrow \boxed{\circ}$ -$\circ$- - -$\circ \Rightarrow \circ$ omitted $\boxed{\circ} \cong A_1 \oplus B_n$,$\circledcirc$ is highest root.

$\alpha_0 \quad \alpha_1 \qquad\qquad \alpha_{n+1}$

$\circledcirc$ => $\boxed{\circ}$ -$\circ$ $-$ $-$ $\circ$ $-\circ$ omitted $\boxed{\circ} \cong A_1 \oplus D_n$, $\circledcirc$ is highest root.

α_0 α_1 α_n

$\circ$ α_{n+1}

Thus we get two bisymmetric superspaces as follows;

$T_\delta = B(n,1)/(A_1 \oplus B_n)$ (the dimension is $2(2n+1)$),

$T_\delta = D(n,1)/(A_1 \oplus D_n)$ (the dimension is $2(2n)$).

Therefore, for $B(n,1)$ if $n = 3m+1$, $m = 3l+1$, $l = 3k+1$, then we have

$T_\delta > T_\gamma > T_\beta > T_\alpha$ and $dim\ T_\delta = 3dim\ T_\gamma = 9dim\ T_\beta = 27dim\ T_\alpha$.

Also, for the case of $D(n,1)$, by $T_\delta = D(n,1)/(A_1 \oplus D_n)$, if $n = 2m$, $m = 2l$, $l = 2k$, then similarly we have

$T_\delta > T_\gamma > T_\beta > T_\alpha$ and $dim\ T_\delta = 2dim\ T_\gamma = 4dim\ T_\beta = 8dim\ T_\alpha$.

(######) $A(n-1,1) = sl(n,2)$ type (the dimension is (n^2+4n+3)):

Let V be a vector space of the matrix set of $Mat(1,2n;\Phi)$ and with an anti-symmetric bilinear form $< x,y > \in \Phi^*$, $\forall x,y \in V$, similarly to section 3, denote by $A := M(V) = \{\begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} | \mathbf{a} \in V = Mat(1,2n;\Phi)\}$.

Then we may define a triple product as follows;

(♠♠♠♠) $< x_1x_2x_3 >=$

$$x_1 \circ (Px_2 \circ x_3) - x_3 \circ (Px_2 \circ x_1) + Px_2 \circ (x_1 \circ x_3),\ \forall x_i\ (i = 1,2,3) \in A.$$

Theorem 4.2 *The triple system A defined by (♠♠♠♠) is a balanced Freudenthal-Kantor triple system with $\varepsilon = -1$ and $\delta = -1$.*

Indeed, by straightforward calculations, we have the relatins $L(x,y) + L(y,x) = K(x,y) =< x,y > Id$ and Eq.(4), these imply that$A$ is a $(-1,-1)$-Freudenthal-Kantor triple system equipped with $dim_\Phi \{K(x,y)\}_{span} = 1$. Hence, from this result of $L_{-1} = A$ (balanced triple system), the standard embedding Lie superalgebra $L(A)$ is obtained $A(n-1,1) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ type, so we have $L_{-1} \oplus L_1 = T(A)$ (anti-Lie triple system), and $L_{-2} \oplus L_0 \oplus L_2 = Der\ T(A)$ (the derivation of $T(A)$). The dimensions are $dim\ L(A) = n^2+4n+3$, $dim\ L_{-1} = dim\ L_1 = 2n$, $dim\ L_{-2} = dim\ L_2 = 1$, and $dim\ L_0 = dim\ A_{n-1} + dim\ (gl(1) \oplus gl(1)) = n^2+1$.

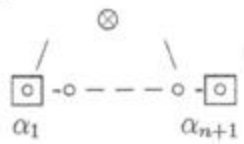
Thus, we can obtain bisymmetric superspaces associated with the balanced triple system as follows;

If $n = 2m,\ m = 2l,\ l = 2k,$

$T_\delta = A(n-1,1)/(A_1 \oplus A_{n-1} \oplus gl(1)), \cdots, T_\alpha = A(k-1,1)/(A_1 \oplus A_{k-1} \oplus gl(1))$

and $T_\delta > T_\gamma > T_\beta > T_\alpha$, furthermore, $dim\ T_\delta = 2dim\ T_\gamma = 4dim\ T_\beta = 8dim\ T_\alpha$.

The extended Dynkin diagram of corresponding $Der\ T(A)$ is the following.



⊗ is the highest root
omitted $\boxed{\circ} \cong A_{n-1} \oplus A_1 \oplus gl(1) = Der\ T(A)$

We call a *quadratic bisymmetric superspace's series* $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ with respect to the bisymmetric superspace $T(A)$ associated with the balanced $(-1,-1)$- Freudenthal-Kantor triple system defined by (♠♠♠) of Theorem 4.1 and (♠♠♠♠) of Theorem 4.2.

Note tht we do not know a construction of the $C(n+1)$ type with the dimension of $n^2 + 5n + 1$ from balanced (-1,-1)-Freudenthal-Kantor triple system, however we can obtain a construction a (1,-1)-Jordan triple system ([42], [44]).

In final of this section, with respect to the bisymmetric superspaces induced from Lie superalgebras $F(4)$ and $G(3)$ types, to describe it breafly, we need the following results. According to ([7], [26]), for the construction of simple Lie superalgebras $F(4)$ with 40 dimension and $G(3)$ with 31 dimension, based upon balanced (-1,-1)-FKTS U equipped with $dim\ U = 8$ (the case of $F(4)$) and $dim\ U = 7$ (the case of $G(3)$), furtheremore, by means of concept of the anti-Lie triple system $T(U) = U \oplus U$ associated with $U(= L_{-1})$, and of the derivation $Der\ T(U) = L_{-2} \oplus L_0 \oplus L_2$, we have studied for the construction of 5-graded Lie superalgebra $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ of following two types;

$F(4)$; ◎ ⇛ $\boxed{\otimes}$–○ <= ○–, $\boxed{\otimes}$ omitted $\cong A_1 \oplus B_3 = Der\ T(U)$, ◎ is the highest root.

$G(3)$; ◎ ≡> $\boxed{\otimes}$–○ ⇚ ○, $\boxed{\otimes}$ omitted $\cong A_1 \oplus G_2 = Der\ T(U)$, ◎ is the highest root.

Here, the figures are obtained from the extended Dynkin diagram of $F(4)$ and $G(3)$ types.

That is, the former type is induced from the octonion algebra $\mathbf{O}$ over the field Φ and the latter type is induced from $Im\ \mathbf{O} = \{x \in \mathbf{O} |\ trace\ x = 0\}$ over Φ, because with respect to the triple product in $U = \mathbf{O}$ or $Im\ \mathbf{O}$, there is the structure of a balanced (-1,-1)-FKTS in $U = L_{-1}$ and $T(U) = L_{-1} \oplus L_1$ is an anti-LTS, hence we can obtain the standard embedding Lie superalgebras $L(U) = T(U) \oplus Der\ T(U)$ equipped with $dim\ L_{-2} = dim\ L_2 = 1$ satisfying $L(U) = F(4)$ (if $dim\ U = 8$) and $L(U) = G(3)$ (if $dim\ U = 7$).

From these results, one can create bisymmetric superspaces $(T_\beta,\ T_\gamma,\ T_\delta)$ as follows;

- $osp(2|2)/(A_1 \oplus gl(1)) < osp(1|4)/(A_1 \oplus A_1) < F(4)/(A_1 \oplus B_3)$,
- $osp(1|2)/A_1 < osp(3|2)/(A_1 \oplus A_1) < G(3)/(A_1 \oplus G_2)$,

where these mean that $osp(1|4) = B(0,2)$, $osp(2|2) = C(2)$, $osp(3|2) = B(1,1)$ and $osp(1|2) = B(0,1)$, also, these dimensions are $dim\ B(0,2) = 14$, $dim\ C(2) = 8$, $dim\ B(1,1) = 12$, and $dim\ B(0,1) = 5$, respectively.

Hereby, the construction of $osp(1|4)$ (resp. $osp(3|2)$) is induced from a balanced (-1,-1)FKTS in the quaternion algebra $\mathbf{H}$ (resp. $Im\ \mathbf{H}$) over Φ, and the construction of $osp(2|2)$ (resp. $osp(1|2)$) is induced from a balanced (-1,-1)FKTS in the separable quadratic field $\mathbf{3}$ (resp. $Im\ \mathbf{3}$) over Φ. We remark for these cases that if $dim\ U = 8$, then $4dim\ T_\beta = 2dim\ T_\gamma = dim\ T_\delta$, and if $dim\ U = 7$, then $2(dim\ T_\beta + 1) = dim\ T_\gamma$, and $2(dim\ T_\gamma + 1) = dim\ T_\gamma$.

Moreprecisely describing for these cases, we have extended Dynkin diagrams as follows;

$B(0,2)$; $\circledcirc \Rightarrow \boxed{\circ} \Rightarrow \bullet$, $\boxed{\circ}$ omitted $\cong A_1 \oplus A_1$, $\circledcirc$ is the highest root,

$C(2)$; $\otimes \Leftarrow \boxed{\otimes}$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus gl(1)$, $\otimes$ is the highest root,

$B(1,1)$; $\circledcirc \Rightarrow \boxed{\otimes} \Rightarrow \circ$, $\boxed{\otimes}$ omitted $\cong A_1 \oplus A_1$, $\circledcirc$ is the highest root, and

$B(0,1)$; $\circledcirc \Rightarrow \boxed{\otimes}$, $\boxed{\otimes}$ omitted $\cong A_1$, $\circledcirc$ is the highest root.

5 Other bisymmetric (super)spaces associated with (ε, δ)-FKTS $U(\varepsilon, \delta)$

In this section, we deal with several bisymmetric (super) spaces without possessing the balanced property. That is, there is a lot of the construction

of Lie (super)algebras associated with triple systems without using balanced properties, for example, we have typical four types as follows:

1) $(\varepsilon, \delta)-$ Jordan triple systems, in particular, $\varepsilon = -1$, *and* $\delta = \pm 1$,

2) $< xyz >=< y, z > x$, where $< x, y >= -\varepsilon < y, x >$,

3) $< xyz >= x^t yz + \delta(z^t yx - z^t xy)$, $x, y, z \in Mat(n, n; \Phi)$, where ${}^t x$ is the transpose matrix of x,

4) $(-1, 1)$-Freudenthal-Kantor triple system induced from $\mathbf{O} \otimes \mathbf{O}$.

1)(a); Let U be a set of $Mat(1, p; \Phi)$. Then $(U, \{xyz\})$ is a Jordan triple system with respect to the product$\{xyz\} =< x, y > z+ < y, z > x- < z, x > y$, where $< x, y >=< y, x > \in \Phi^*$ and so the standard embedding Lie algebra $L(U) = L_{-1} \oplus L_0 \oplus L_0$ associated with $U = L_{-1}$ is a B_{n+1} type (if $p = 2n + 1$) and a D_{n+1} (if $p = 2n$), because $K(x, y) = 0$, $\forall x, y \in U$.

The extended Dynkin diagram is the following;

$\boxed{\circ}$-$\circ - \circ -$ $-\circ =>$ $\circ$, omitted $\boxed{\circ} \cong B_n \oplus gl(1) = Der\ T(U)$.

α_1 $\quad\quad\quad\quad\quad\quad\quad\quad$ α_{n+1}

$\quad \circledcirc\ \alpha_0$

$\boxed{\circ}$-$\circ - \circ -$ $- \circ - \circ$, omitted $\boxed{\circ} \cong D_n \oplus gl(1) = Der\ T(U)$.

α_1 $\quad\quad\quad\quad\quad$ α_n

$\quad \circledcirc\ \alpha_0 \quad\quad\quad \circ\ \alpha_{n+1}$

Thus we can obtain for the standard embedding Lie algebra of two types. That is, $L(U) = B_{n+1}$ **if** $p = 2n + 1$ and $L(U) = D_{n+1}$ **if** $p = 2n$, and $L_{-2} = L_2 = 0$, because $K(x, y) = 0$ (identically).

Hence the bisymmetric spaces of these cases are given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$\cdot\ T_\alpha = B_{k+1}/(B_k \oplus gl(1)) < T_\beta = B_{l+1}/(B_l \oplus gl(1)) < T_\gamma = B_{m+1}/(B_m \oplus gl(1)) < T_\delta = B_{n+1}/(B_n \oplus gl(1))$, and 27 $dim\ T_\alpha = 9\ dim\ T_\beta = 3\ dim T_\gamma = dim\ T_\delta$, where $n = 3m + 1, m = 3l + 1,$ and $l = 3k + 1$.

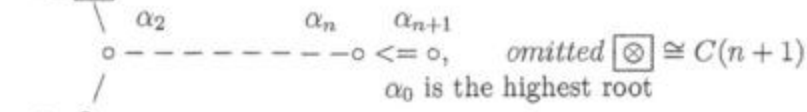
Similarly, for D_{n+1} type, we have

$(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying

$\cdot\ T_\alpha = D_{k+1}/(D_k \oplus gl(1)) < T_\beta = D_{l+1}/(D_l \oplus gl(1)) < T_\gamma = D_{m+1}/(D_m \oplus gl(1)) < T_\delta = D_{n+1}/(D_n \oplus gl(1))$, and 8 $dim\ T_\alpha = 4\ dim\ T_\beta = 2\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2l,$ and $l = 2k$.

1)(b); Let U be a set of $Mat(1, 2n; \Phi)$. Then $(U, \{xyz\})$ is an anti-Jordan triple system ($\varepsilon = 1$ and $\delta = -1$) with respect to the product$\{xyz\} =< x, y > z+ < y, z > x- < z, x > y$, where $< x, y >= - < y, x > \in \Phi^*$ and so

the standard embedding Lie algebra $L(U) = L_{-1} \oplus L_0 \oplus L_0$ associated with $U = L_{-1}$ is a $C(n+1)$ type with $dim\ L(U) = 2n^2 + 5n + 1$.
The extended Dynkin diagram is the following;

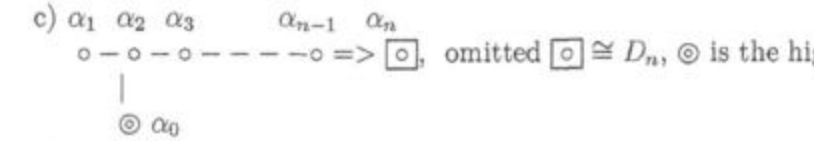


Note that $L(U) = L_{-1} \oplus L_0 \oplus L_1$, $Der\ T(U) = L_0 \cong C_n \oplus gl(1)$, and $L_{-2} = L_2 = 0$, because $K(x, y) = 0$ identically.
Thus the bisymmetric superspaces of this case are given by $(T_\alpha, T_\beta, T_\gamma, T_\delta)$ satisfying
$\cdot\ T_\alpha = C_{k+1}/(C_k \oplus gl(1)) < T_\beta = C_{l+1}/(C_l \oplus gl(1)) < T_\gamma = C_{m+1}/(C_m \oplus gl(1)) < T_\delta = C_{n+1}/(C_n \oplus gl(1))$, and 8 $dim\ T_\alpha = 4\ dim\ T_\beta = 2\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2l$, and $l = 2k$.

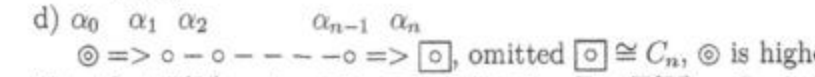
2) Let U be a set of $Mat(1, n; \Phi)$. Then $(U, \{xyz\})$ is a $(-1, \delta)$-FKTS with respect to the product$\{xyz\} =< y, z > x$, where $< x, y >=< y, x > \in \Phi^*$ and so the standard embedding Lie algebra ($\delta = 1$) is 5-graded B_n type (this is called (2)(c) type), and the standard embeding Lie superalgebra ($\delta = -1$) is 5-graded $B(0, n)$ type (this is called (2)(d) type), That is, $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ associated with $U = L_{-1}$ is two cases as follows;
$dim\ L(U) = n(2n+1)$ *if* $\delta = 1$, *and* $K(x, z)y =< z, y > x - < x, y > z$,
$dim\ L(U) = 2n^2 + 3n$ *if* $\delta = -1$, *and* $K(x, z)y =< y, x > z + < y, z > x$

The extended Dynkin diagrams are the following;

c)



d)



Note that $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $Der\ T(U) = L_{-2} \oplus L_0 \oplus L_2 \cong D_n$ (if $\delta = 1$ of the case (c)) and $Der\ T(U) = L_{-2} \oplus L_0 \oplus L_2 \cong C_n$ (if $\delta = -1$ of the case (d)).
The bisymmetric (super)spaces of these cases (c) and (d) are given by

$(T_\alpha,\ T_\beta,\ T_\gamma,\ T_\delta)$ satisfying
c) $T_\alpha = B_k/D_k < T_\beta = B_l/D_l < T_\gamma = B_m/D_m < T_\delta = B_n/D_n$, and $8\ dim\ T_\alpha = 4\ dim\ T_\beta = 2\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2k,$ and $l = 2k$.
Similarly, for $B(0,n)$ we have
d) $T_\alpha = B(0,k)/C_k < T_\beta = B(0,l)/C_l < T_\gamma = B(0,m)/C_m < T_\delta = B(0,n)/C_n$, and $8\ dim\ T_\alpha = 4\ dim\ T_\beta = 2\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2l,$ and $k = 2l$.

For the case of which contains $\varepsilon = 1$, we can generalize to the following.
Remark For the bilinear form $< x,y >= -\varepsilon < y,x > \in \Phi^*$, and introduce the product $\{xyz\} =< y,z > x$, then this triple system is a (ε,δ)-Freudenthal-Kantor triple system without balanced property.

3) (e); For the case of $\delta = 1$, let U be aset of $Mat\ (2n+1, 2n+1 : \Phi)$. Then $(U, \{xyz\})$ is a structurable algebra (or a (-1,1)-FKTS, [1], [31], [32], [33] and references of therein) with respect to the product $\{xyz\} = x^t yz + z^t yx - z^t xy,\ \forall \in U$. Then the Lie algebra $L(U)$ is a 5-graded B_{3n+1} type with $18n^2 + 15n + 3$ dimension. The extended Dynkin diagram is the following;

$\circ - \circ - \circ - \ - \circ - \boxed{\circ} - \circ - - - \circ => \circ$, $\boxed{\circ}$ omitted $\cong D_{2n+1} \oplus B_n = Der\ T(U)$,

α_1 | α_{2n+1} α_{3n+1}
◎ α_0 α_0 (◎) is the highest root.

Next, we consider the case 3) (d)'s case, let U be a set of $Mat\ (2n, 2n; \Phi)$, then the Lie algebra $L(U)$ is a 5-graded D_{3n} type with $18n^2 - 3n$ dimension. The extended Dynkin diagram is the following;

$\circ - \circ - \circ - \ - \circ - \boxed{\circ} - \circ - - - \circ - \circ$, $\boxed{\circ}$ omitted $\cong D_{2n} \oplus D_n = Der\ T(U)$,

α_1 | α_{2n} | α_{3n-1}
◎ α_0 ∘ α_{3n} α_0 (◎) is the highest root.

Hence the bisymmetric spaces of the case (e) is given by $(T_\alpha,\ T_\beta,\ T_\gamma,\ T_\delta)$ satisfying if $dim\ U = (2n+1)^2$, $Der\ T(U) = D_{2n+1} \oplus B_n$,
· $T_\alpha = B_{k+1}/(D_{2k+1} \oplus B_k) < T_\beta = B_{3l+1}/(D_{2l+1} \oplus B_l) <$
$T_\gamma = B_{3m+1}/(D_{2m+1} \oplus B_m) < T_\delta = B_{3n+1}/(D_{2n+1} \oplus B_n)$, and $729\ dim\ T_\alpha = 81\ dim\ T_\beta = 9\ dim T_\gamma = dim\ T_\delta$, where $n = 3m+1, m = 3l+1,$ and $l = 3k+1$.
Similarly, for D_{3n} type if $dim\ U = (2n)^2$, $Der\ T(U) = D_{2n} \oplus D_n$, we have $(T_\alpha,\ T_\beta,\ T_\gamma,\ T_\delta)$ satisfying

$\cdot\ T_\alpha = D_{3k}/(D_{2k} \oplus D_k) < T_\beta = D_{3l}/(D_{2l} \oplus D_l) < T_\gamma = D_{3m}/(D_{2m} \oplus D_m) < T_\delta = D_{3n}/(D_{2n} \oplus D_n)$, and $64\ dim\ T_\alpha = 16\ dim\ T_\beta = 4\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2l$, and $l = 2k$. Note that for 5-graded cases, $T(U) = L_{-1} \oplus L_1$, $Der\ T(U) = L_{-2} \oplus L_0 \oplus L_2$ and $T(U) = L(U)/Der\ T(U)$.

3) (f); For the case of $\delta = -1$, let U be a set of $Mat\ (2n+1, 2n+1; \Phi)$. Then $(U, \{xyz\})$ is a $(-1,-1)$−FKTS (or an anti-structurable algebra ([31]) with respect to the product

$$\{xyz\} = x^t yz - z^t yx + z^t xy, \quad \forall x, y, z \in U.$$

Then the Lie superalgebra constructed from $U = L_{-1}$ is 5-graded $B(n, 2n+1)$ type with $(18n^2 + 19n + 5)$ dimension.

The extended Dynkin diagram is the following;

$\circledcirc => \circ - \circ - \ -\circ - \boxed{\circ} - \circ - - -\circ => \circ$, $\boxed{\circ}$ omitted $\cong C_{2n+1} \oplus B_n$,

$\alpha_0 \quad \alpha_1 \qquad\qquad \alpha_{2n+1} \qquad\qquad \alpha_{3n+1}$

Hence we remark that $L_0 = \{L(x,y) | x, y \in U\}_{span} \cong A_{2n} \oplus B_n \oplus gl(1)$.

Next let U be a set of $Mat(2n, 2n; \Phi)$. Then similarly we have the fact that $(U, \{xyz\})$ is a $(-1,-1)$−FKTS (or an anti-structurable algebra ([31]) with respect to the product

$$\{xyz\} = x^t yz - z^t yx + z^t xy, \quad \forall x, y, z \in U.$$

Then the Lie superalgebra constructed from $U = L_{-1}$ is 5-graded $D(n, 2n)$ type with $(18n^2 + n)$ dimension.

The extended Dynkin diagram is the following;

$\circledcirc => \circ - \circ - \ -\circ - \boxed{\circ} - \circ - - - \circ - \circ$, $\boxed{\circ}$ omitted $\cong C_{2n} \oplus D_n = Der\ T(U)$,

$\alpha_0 \quad \alpha_1 \qquad\qquad \alpha_{2n} \qquad\qquad | \quad \alpha_{3n-1}$

$\qquad\qquad\qquad

$\cdot\ T_\alpha = D(k,2k)/(C_{2k} \oplus D_k) <\ T_\beta = D(l,2l)/(C_{2l} \oplus D_l) <$
$T_\gamma = D(m,2m)/(C_{2m} \oplus D_m) <\ T_\delta = D(n,2n)/(C_{2n} \oplus D_n)$, and 64 $dim\ T_\alpha =$ 16 $dim\ T_\beta = 4\ dim T_\gamma = dim\ T_\delta$, where $n = 2m, m = 2l,$ and $l = 2k$. Note that for these 5-graded cases, $T(U) = L_{-1} \oplus L_1$, $Der\ T(U) = L_{-2} \oplus L_0 \oplus L_2$ and $T(U) = L(U)/Der\ T(U)$.

Example 5.1 (a generalization of δ structurable algebras) For the case of $U = Mat(m,n;\Phi)$ $(if\ m \neq n)$, note that the triple product

$$\{xyz\} = x^t yz + \delta(z^t yx + z^t xy), \quad \forall x,y,z \in U.$$

is a (-1,δ)-FKTS, of cause, there is the both cases of $\delta = \pm 1$.

4) (g); Let U be a vector space's set of $\mathbf{O} \otimes \mathbf{O}$ (the dimension of U is 64). Then $(U, < xyz >)$ is a (-1,1)-FKTS with respect to the product

$$< xyz > = (x\overline{y})z + (z\overline{y})x - (z\overline{x})y, \ \forall\ x,y,z \in U,$$

where $\overline{x}$ is the involution satisfying $\overline{xy} = \overline{yx}$, and so the standard embedding Lie algebra $L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is E_8 type with $U = L_{-1}$, $dim\ L_{-2} = dim\ L_2 = 14$ *and* $dim\ L_0 = 92$. This (-1,1)-FKTS is called a structurable algebra or for more general case, it is said to be a normal triality algebra ([1], [37]).

For subalgebras $A_1$and A_2 of $\mathbf{O}$, if we use the notation $A = A_1 \otimes A_2$, $dim\ A_1$, $dim\ A_2$, then the Lie algebras obtained from these subalgebras are as follows:

$L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \cong E_8$, $L_0 \cong D_7 \oplus gl(1)$, $L_{-2} \oplus L_0 \oplus L_2 \cong D_8$,

$dim A_2 \backslash dim A_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

For the case's E_8, considering the extended Dynkin diagram, we have

$L_{-1} \oplus L_1 \cong L(U)/(L_{-2} \oplus L_0 \oplus L_2) = E_8/D_8$ with dim $(L_{-1} \oplus L_1)$=128;

From these results, we obtain the bisymmetric space's series of tensor type, which is called a *tensor type bisymmetric space,*

$$(\heartsuit)\quad F_4/B_4 < E_6/(D_5 \oplus gl(1)) < E_7/(D_6 \oplus A_1) < E_8/D_8,$$

and the dimensions of bisymmetric spaces $(\heartsuit)$ of tensor type are

$$16,\quad 32,\quad 64,\quad 128.$$

With respect to extended Dynkin diagrams (symmetric spaces) of E_7, E_6, F_4:

$\circ\!-\!\boxed{\circ}\!-\circ-\circ-\circ-\circ-\circledcirc$, $\boxed{\circ}$ omitted $\cong D_6 \oplus A_1$, and $\circledcirc$ is the highest root.

$$\begin{array}{ccccccccc} & & & & \circ & & & & \\ \boxed{\circ} & - & \circ & - & \circ & - & \circ & - & \boxed{\circ} \\ & & & & \circledcirc & & & & \end{array}$$

$\boxed{\circ}$ omitted $\cong D_5 \oplus gl(1)$, and $\circledcirc$ is the highest root.

$$\begin{array}{ccccccc} & & & & \circ & & \\ & & & & \circledcirc & & \\ \circledcirc & - & \circ & - & \circ & = > & \circ - \boxed{\circ} \end{array}$$

$\boxed{\circ}$ omitted $\cong B_4$, and $\circledcirc$ is the highest root.

Remark Let $(U, < xyz >)$ be a (ε, δ)-FKTS. If $J \in End\ U$ satisfies $J^2 = -Id$ and $J < xyz > = < JxJyJz >$, then $(U, \{xyz\})$ is a $(-\varepsilon, \delta)$-FKTS with respect the new product defined by $\{xyz\} = < xJyz >$.

6 A view point of geometric structures of $U(\varepsilon, \delta)$

6.1 A generalized curvature and torsion tensors associated with (ε, δ)-FKTS $U(\varepsilon, \delta)$

Let $L = L(U(\varepsilon, \delta)) = L(W, W) \oplus W$ be the Lie (super)algebra defined from a δ LTS $W = T(\varepsilon, \delta)$ and $L(W, W) = Der\ (T(\varepsilon, \delta), T(\varepsilon, \delta))$ as in section one, that is, the δ-LTS $W = T(\varepsilon, \delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon, \delta)$ (as L_{-1} has the structure of a (ε, δ)-FKTS) over the field Φ of $ch\ \Phi \neq 2, 3$.

We introduce a generalization of covariant derivative ∇ in differential geometry as follows; $\nabla : L \to \text{End } L$ defined by

$\nabla_X Y = [X, Y] = -\delta[Y, X]$,
$\nabla_X [Y, Z] = [YZX] = -\delta[ZYX]$,
$\nabla_{[X,Y]} Z = -[XYZ] = -\delta[YXZ]$,
$\nabla_{[X,Y]}[V, Z] = [[V, Z][X, Y]] = -\delta[[X, Y][V, Z]]$,
for any $X, Y, Z \in W$.

Furtheremore, a generalized curvature tensor defined by

$$C_\delta(X, Y) = \nabla_X \nabla_Y - \delta \nabla_Y \nabla_X - \nabla_{[X,Y]} \tag{12}$$

is identically zero, i.e., $C_\delta(X, Y) = 0$ in L, for any $X, Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$\begin{aligned} C_\delta(X, Y)Z &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)Z - \nabla_{[X,Y]} Z \\ &= \nabla_X [Y, Z] - \delta \nabla_Y [X, Z] + [XYZ] \\ &= [YZX] - \delta[XZY] + [XYZ] \\ &= [YZX] + [ZXY] + [XYZ] = 0. \end{aligned}$$

Second, it follow

$$\begin{aligned} C_\delta(X, Y)[V, Z] &= (\nabla_X \nabla_Y - \delta \nabla_Y \nabla_X)[V, Z] - \nabla_{[X,Y]}[V, Z] \\ &= [X, [VZY]] - \delta[Y, [VZX]] + \delta[[X, Y], [V, Z]] \\ &= [X, L(V, Z)Y] - \delta[Y, L(V, Z)X] - L(V, Z)[X, Y] = 0 \end{aligned}$$

(by $[Y, L(V, Z)X] = -\delta[L(V, Z)X, Y]$ and $[[X, Y], [V, Z]] = -\delta[[V, Z], [X, Y]]$) for any $X, Y, Z, V \in T(\varepsilon, \delta)$.

However a generalized torsion tensor defined by

$$S_\delta(X, Y) = \nabla_X Y - \delta \nabla_Y X - [X, Y] \tag{13}$$

is not zero, since it gives $S_\delta(X, Y) = [X, Y] - \delta[Y, X] - [X, Y] = [X, Y]$.

To remember the J in section one, we have defined the Nijenhuis operator

$$N(X, Y) = [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY],$$

where for the case of $\varepsilon\delta = 1$, J is an almost complex structure on W, this concept (the case of $\delta = 1$) is appeared in ([38]). If we now set $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we have $J^2 = -Id$, or $J^2 = Id$, respectively, hence these concept mean that there is a comlex or para complex structures in the symmetric (super)space W, and furtermore, by means of the treatment of a wide choice for J, there is a structure of polarized, invariant, straight, twisted, etc. as well as ([5], [6]).

6.2 Metric structures associated with (ε, δ)-FKTS $U(\varepsilon, \delta)$

In this section, we will introduce a certain metric structure with respect to the (ε, δ)-FKTS $U(\varepsilon, \delta)$ over the field Φ of $ch\ \Phi \neq 2, 3$ as follows (hereby, it is said to be a *Killing form* of the triple system $U(\varepsilon, \delta)$);

$$\gamma(x, y) = \frac{1}{2} Trace[2(R(x, y) - \varepsilon R(y, x)) + \delta(\varepsilon L(x, y) - L(y, x))],$$

this bilinear form $\gamma(x, y)$ is a map: $U(\varepsilon, \delta) \times U(\varepsilon, \delta) \longrightarrow \Phi$ and denote by $L(x, y)z =< xyz >$ and $R(x, y)z =< zxy >$.

Next we can define the Killing form of $T(\varepsilon, \delta)$ associated with $U(\varepsilon, \delta)$;

$$\alpha(X, Y) = \frac{1}{2} Trace(R(X, Y) + \delta R(Y, X)),$$

and by using Eq.(10) in section one, we obtain $\alpha(X, Y) = \gamma(b_1, a_2) + \delta\gamma(b_2, a_1)$, where $X = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, Y = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$.

Furtheremore, for $\delta = \pm 1$, we have

$\gamma(< xyz >, w) + \delta\gamma(z, < yxw >) = 0,\ \gamma(< xyz >, w) + \delta\gamma(x, < wzy) >= 0,$

$$\gamma(Dx, y) + \gamma(x, Dy) = 0,\ for\ all\ D \in Der\ U(\varepsilon, \delta)$$

$$\alpha(X, Y) = \beta(Y, X) = \delta\beta(X, Y), \forall\ X, Y \in T(\varepsilon, \delta) \subset L(U(\varepsilon, \delta)),$$

where $\beta(X, Y) = strace\ ad\ X\ ad\ Y$, (the Killing form of Lie superalgebras). Indeed, from relations $gL(x, y)g^{-1} = L(gx, gy)$, $gR(x, y)g^{-1} = R(gx, gy)$, $[D, L(x, y)] = L(Dx, y) + L(x, Dy)$ and $[D, R(x, y)] = R(Dx, y) + R(x, Dy)$, it follows.

On the other hand, for special balanced cases of $U = U(\varepsilon, \delta)$, by straightforward calculations, we obtain the relation;

$$\gamma(x,y) = \frac{\delta}{2} < x,y > dim\ U + (1-\delta) < x,y > .$$

These results are appeared for a part study in ([14], [15], [18], [37]). However, we do not go the details.

6.3 A characterization of automorphisms of $U(\varepsilon, \delta)$

In this section, we will discuss with a certain automorpism of $T(\varepsilon, \delta)$ and Lie (super)algebras $L(U(\varepsilon, \delta))$ associated with an automorphism of (ε, δ)-Freudenthal-Kantor triple systems $U(\varepsilon, \delta)$ over the field Φ of $ch\ \Phi \neq 2, 3$.

Proposition 6.1 *Let $U(\varepsilon, \delta)(= U)$ be a (ε, δ)-Freudenthal-Kantor triple system, the Killing form $\gamma\ (x,y)$ of U and g be any automorphism of the triple system U. Then we have*

$$\gamma\ (x,y) = \gamma\ (gx, gy), \quad for\ all\ x, y \in U$$

Furthermore, $\gamma(S(x,y)z, w) + \gamma(z, S(x,y)w) = 0, \quad for\ all\ x, y, z, w \in U$, where $S(x,y) := L(x,y) + \varepsilon L(y,x) \in Der\ U$.

Indeed, from $gR(x,y)g^{-1} = R(gx, gy)$ and $gL(x,y)g^{-1} = L(gx, gy)$, it follows.

It seems that this relation of $S(x,y)$ means a certain adjoint representation with respect to the $U(\varepsilon, \delta)$

Next, for any element $g \in Epi(U(\varepsilon, \delta))$, we define an endomorphism $\tilde{g}$ of $L(U(\varepsilon, \delta))$ as follows; $(Der\ T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta) \longrightarrow$
$\begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix} (Der\ T(\varepsilon, \delta)) \begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix}^{-1} \oplus \begin{pmatrix} 0 & \delta g \\ -\varepsilon g & 0 \end{pmatrix} (T(\varepsilon, \delta))$.

In particular, if $g = Id$, then $\tilde{g} = \begin{pmatrix} 0 & \delta Id \\ -\varepsilon Id & 0 \end{pmatrix}$ is an automorphism of $L(U(\varepsilon, \delta))$.

Hence, under these notations, by straightforward calculations, we have the following.

Theorem 6.2 *The following are equivalent;*

i) g is a linear transformation of $U(\varepsilon,\delta)$ satisfying $g < xyz >=< gxgygz >$,
ii) $\tilde{g}$ is an automorphism of $T(\varepsilon,\delta)$,
iii) $\tilde{g}$ is an automorphism of $L(U(\varepsilon,\delta))$.
Indeed, from $\tilde{g}[XYZ] = [(\tilde{g}X)(\tilde{g}Y)(\tilde{g}Z)]$, $\forall X,Y,Z \in T(\varepsilon,\delta)$, it follows.
Remark Note that for the Killing forms $\alpha(X,Y)$, $\beta\ (X,Y)$, we have $\beta\ (\tilde{g}X,\tilde{g}Y) = \beta\ (X,Y)$, and $\alpha\ (\tilde{g}X,\tilde{g}Y) = \alpha\ (X,Y)$, $\forall X,Y \in T(\varepsilon,\delta) \subset L(\varepsilon,\delta)$.

Concluding Remark

–In homage of Professor Susumu Okubo (1930-2015)–
he is a Wigner Medal Winner (2006) and a Nishina memorial prize (1976).

For relationships between triple systems and mathematical physics, we would like to refer S. Okubo's book ([60]) and ([23], [27], [39], [40], [41], [62]), and for a certain history in our subjects of nonassociative algebras, refer ([44]). In final comments of this paper, with respect to the nonassociative algebras (in particular, Lie admissible algebras), we note that there are a lot of important works due to Prof. R. Santilli for mathematical physics into Hadronic Journal's articles.

Appendix I (the case of $\varepsilon = 1 and\ \delta = 1$)

In the author's judgement, roughly speaking, there is a reason for studying triple systems as follows.

In this appendix, we note that there is a certain correspondence of our Lie algebra $L(U)$ and simple Lie algebra $\mathfrak{g}$ over an algebraically closed field Φ of characteristic zero with the heighest root ρ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\triangle$ is the root system, $\triangle^+$ is the positive root system, $\triangle_1 = \{\alpha \in \triangle^+ | \rho \neq \alpha,\ \rho - \alpha \in \triangle^+\}$, $\triangle_0 = \{\alpha \in \triangle^+ | \rho \neq \alpha,\ \rho - \alpha \notin \triangle^+\}$, That is, $\triangle^+ = \triangle_0 \cup \triangle_1 \cup \{\rho\}$. Then we have a decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\triangle} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $dim \mathfrak{g}_\alpha = 1$.

Furthermore we choose certain basis $\{H_\rho, E_{\pm\alpha}, E_{\pm\beta}, E_{-\rho}, E_\rho | \alpha \in \triangle_1, \beta \in \triangle_0\}$ of subspaces $\sum \mathfrak{g}_\alpha$ $(\alpha \in \triangle)$ and $H_\rho \in \mathfrak{h}$ satisfying the relations;
$[E_{-\rho}, E_\rho] = H_\rho, [H_\rho, E_{-\rho}] = -2E_{-\rho}, [H_\rho, E_\rho] = 2E_\rho$, $[E_\rho, E_{-\alpha}] = E_{\rho-\alpha}$,
$[E_{-\rho}, E_\alpha] = -E_{-\rho+\alpha}$, $(if\ \alpha \in \triangle_1)$, $[E_\rho, E_{\pm\beta}] = 0$, $(if\ \beta \in \triangle_0)$.
Note that $< E_{-\rho}, E_\rho, H_\rho >_{span} = sl(2), i.e., A_1\ type.$

Then we have a decomposition;

$\mathfrak{g} = \sum_{i=-2}^{2} \mathfrak{g}_i$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $[H_\rho, \mathfrak{g}_i] = i\,\mathfrak{g}_i, i = 0 \pm 1, \pm 2$
where $\mathfrak{g}_{-2} = \mathfrak{g}_{-\rho} =< E_{-\rho} >_{span}$, $\mathfrak{g}_2 = \mathfrak{g}_\rho =< E_\rho >_{span}$, $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, $\mathfrak{g}_{-1} = \sum_{\alpha \in \Delta_1} \mathfrak{g}_{-\alpha}, \mathfrak{g}_1 = \sum_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$, note that for $ad\ H_\rho$, this $\mathfrak{g}_i$ is the vector space of the eigen value i, because $ad\ H_\rho \mathfrak{g}_i = i\mathfrak{g}_i$ by $[H_\rho, E_\alpha] = E_\alpha$ (if $\alpha \in \Delta_1$) $[H_\rho, E_\beta] = 0$ (if $\beta \in \Delta_0$), $\mathfrak{g}_1 =< E_\alpha | \alpha \in \Delta_1 >_{span}$, etc..
By means of notation being as above, a little long calculations, we can verify the following.

Theorem A.1. *Let $\mathfrak{g}$ be a simple Lie algebra and for the vector space $\mathfrak{g}_{-1}$ of $\mathfrak{g}$, then $U = \mathfrak{g}_{-1}$ is a balanced $(1,1)$-Freudenthal-Kantor triple system with respect to the triple product*
$L(x,y)z =:< xyz >= [[x,[y,E_\rho]],z]$, $\forall x, y, z \in \mathfrak{g}_{-1}$, *equipped with* $K(x,y) = -L(x,y) + L(y,x) =< x,y > Id$, *where the bilinear form* $< x,y >$ *is defind by* $[x,y] =< x,y > E_{-\rho}$.
Furthermore, for the standard embedding Lie algebra $L(U)$ associated with $U = \mathfrak{g}_{-1}$ and the Lie triple system $T(1,1) = U \oplus U$ (the case of $\varepsilon = 1, \delta = 1$) in section one, we have an isomorphism of Lie algebras as follows.

Theorem A.2. *For Lie algebras $L(U)$ and $\mathfrak{g}$ as above, we have*
$L(U) \simeq \mathfrak{g}$ *as the Lie algebras isomorphism, by an isomorphism mapping* ϕ,
$\phi : \ L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$,

$$D(T(1,1),T(1,1)) := \left\{ \begin{pmatrix} L(x,y) & Id \\ -Id & L(y,x) \end{pmatrix} \right\}_{span} \longrightarrow E_{-\rho} + ad([x,[y,E_\rho]]) + E_\rho,$$

$$T(1,1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U \right\}_{span} \longrightarrow x + [y, E_\rho].$$

where $< x,y > E_{-\rho} = [x,y]$, $\forall x, y \in U$. *Note that* $< x,y > E_\rho = [[x,E_\rho],[y,E_\rho]]$.

Appendix II (the case of $\varepsilon = -1 and\ \delta = 1$). In this appendix II, we describe another correspondence of $L(U)$ and $\mathfrak{g}$.

For simple Lie algebras $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and the root system Δ, then we have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
And we intoroduce notations as follows;
the heighest root ρ ($\rho = 2\alpha_1 + \sum_{i \neq 1} n_i \alpha_i$), α_i is the simple roots,
$\Delta_0^* = \{\alpha \in \Delta | \alpha = \sum_{i \neq 1} m_i\ \alpha_i,\ m_i$: nonnegative integer$\}$,
$\Delta_1^* = \{\alpha \in \Delta | \alpha = \alpha_1 + \sum_{i \neq 1} k_i\ \alpha_i,\ k_i$: nonnegative integer$\}$,

$\triangle_2^* = \{\alpha \in \triangle | \alpha = 2\alpha_1 + \sum_{i\neq 1} l_i\ \alpha_i,\ l_i$: nonnegative integer$\}$.
Note that $\rho \in \triangle_2^*$, however, note that the highest root of A_{n-1} type's case with (n^2-1) dimension is $\rho = \alpha_1 + \alpha_n + \sum_{i\neq 1,n} m_i\ \alpha_i$, where m_i is positive integers.
Then based on these notations, we have a decomposition;

$\mathfrak{g} = \sum_{i=-2}^{2} \mathfrak{g}_i,\ \ [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$,

where $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha\in\triangle_0^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 2} = \sum_{\alpha\in\triangle_2^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha\in\triangle_1^*} \mathfrak{g}_{\pm\alpha}$.
We consider an endomorphism τ of $\mathfrak{g}$ satisfying $\tau^2 = id$ and $\tau([X,Y]) = [\tau(X), \tau(Y)], \forall X, Y \in \mathfrak{g}$, that is, $\tau : \mathfrak{g}_{-i} \to \mathfrak{g}_i$, by straightfoward calculatins, we obtain the following.

Theorem A.3. *The vector space $U = \mathfrak{g}_{-1}$ has a structure of (-1,1)-Freudenthal-Kantor triple system with respect to the triple product $L(x,y)z :=< xyz >= [[x, \tau(y)], z],\ \forall x, y, z \in U$.*

Hence we can verify the following theorem.

Theorem A.4. *For Lie algebras $L(U)$ and $\mathfrak{g}$, we have*
$L(U) \simeq \mathfrak{g}$ as the Lie algebras isomorphism, by an isomorphism mapping ϕ,
$\phi :\ L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$,

$$D(T(-1,1), T(-1,1)) := \left\{ \begin{pmatrix} L(x,y) & K(a,b) \\ K(c,d) & -L(y,x) \end{pmatrix} \right\}_{span} \longrightarrow$$

$$[a,b] + ad([x,\tau(y)]) + [\tau(c), \tau(d)],$$

$$T(-1,1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U \right\}_{span} \longrightarrow x + \tau(y).$$

where $L(x,y)z =< xyz >,\ K(x,y)z =< xzy > - < yzx >,\ \forall x, yz \in U$.

With respect to an example of above τ of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\triangle} \mathfrak{g}_\alpha,\ \ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, note that if $\theta : h \to -h,\ (h \in \mathfrak{h})$, and $E_\alpha \to -E_{-\alpha},\ \ (E_\alpha \in \mathfrak{g}_\alpha)$, then it satisfies $\theta^2 = id$, and $\theta([X,Y]) = [\theta(X), \theta(Y)],\ \forall X, Y \in \mathfrak{g}$.

Appendix III (the case of $\varepsilon = -1 and\ \delta = -1$)

In this appendix, we note that there is a certain correspondence of our Lie superalgebra $L(U)$ and 5-graded simple basic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over an algebraically closed field Φ of characteristic zero with nondegenerate Killing form, (for example, $A(m,n),\ B(m,n),\ D(m,n), G(3), F(4)$), now let the heighest root be ρ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\triangle$ be the root system, $\triangle^+$ be the positive root system, $\triangle_1 = \{\alpha \in \triangle^+ | \rho \neq$

$\alpha,\ \rho-\alpha \in \triangle^+\}$, $\triangle_0 = \{\alpha \in \triangle^+ | \rho \neq \alpha,\ \rho-\alpha \notin \triangle^+\}$, That is, $\triangle^+ = \triangle_0 \cup \triangle_1 \cup \{\rho\}$. Then we have a decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $dim \mathfrak{g}_\alpha = 1$.

Furthermore we choose certain basis $\{H_\rho, E_{\pm\alpha}, E_{\pm\beta}, E_{-\rho}, E_\rho | \alpha \in \triangle_1, \beta \in \triangle_0\}$ of subspaces $\sum \mathfrak{g}_\alpha$ $(\alpha \in \triangle)$ and $H_\rho \in \mathfrak{h}$ satisfying the relations;
$[E_{-\rho}, E_\rho] = H_\rho, [H_\rho, E_{-\rho}] = -2E_{-\rho}, [H_\rho, E_\rho] = 2E_\rho,\ [E_\rho, E_{-\alpha}] = E_{\rho-\alpha}$,
$[E_{-\rho}, E_\alpha] = -E_{-\rho+\alpha},\ (if\ \alpha \in \triangle_1),\ [E_\rho, E_{\pm\beta}] = 0,\ (if\ \beta \in \triangle_0)$.
Note that $< E_{-\rho}, E_\rho, H_\rho >_{span} = sl(2), i.e., A_1\ type.$

Then we have a decomposition;

$\mathfrak{g} = \sum_{i=-2}^{2} \mathfrak{g}_i,\ \ [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},\ [H_\rho, \mathfrak{g}_i] = i\ \mathfrak{g}_i, i = 0 \pm 1, \pm 2$

where $\mathfrak{g}_{-2} = \mathfrak{g}_{-\rho} =< E_{-\rho} >_{span}$, $\mathfrak{g}_2 = \mathfrak{g}_\rho =< E_\rho >_{span}$, $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_0} \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, $\mathfrak{g}_{-1} = \sum_{\alpha\in\Delta_1} \mathfrak{g}_{-\alpha}, \mathfrak{g}_1 = \sum_{\alpha\in\Delta_1} \mathfrak{g}_\alpha$, note that for $ad\ H_\rho$, this $\mathfrak{g}_i$ is the vector space of the eigen value i, because $ad\ H_\rho \mathfrak{g}_i = i\mathfrak{g}_i$ by $[H_\rho, E_\alpha] = E_\alpha$ (if $\alpha \in \triangle_1$) $[H_\rho, E_\beta] = 0$ (if $\beta \in \triangle_0$), $\mathfrak{g}_1 =< E_\alpha | \alpha \in \triangle_1 >_{span}$, and $\mathfrak{g}_0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

By means of notation being as above, a little long calculations, we can verify the following.

Theorem A.5. *Let $\mathfrak{g}$ be a simple Lie superalgebra as avove and for the vector space $\mathfrak{g}_{-1}$ of $\mathfrak{g}$, then $U = \mathfrak{g}_{-1}$ is a balanced $(-1,-1)$-Freudenthal-Kantor triple system with respect to the triple product*
$L(x,y)z =:< xyz >= [[x,[y,E_\rho]],z],\ \forall x,y,z \in \mathfrak{g}_{-1}$, *equipped with* $K(x,y) = L(x,y) + L(y,x) =< x,y >$, *where the bilinear form* $< x,y >$ *is defined by* $[x,y] = - < x,y > E_{-\rho}$.

Furthermore, for the standard embedding Lie superalgebra $L(U)$ associated with $U = \mathfrak{g}_{-1}$ and the anti-Lie triple system $T(-1,-1) = U \oplus U$ (the case of $\varepsilon = -1, \delta = -1$) in section one, we have an isomorphism of Lie algebras as follows.

Theorem A.6. *For Lie superalgebras $L(U)$ and $\mathfrak{g}$ as above, we have $L(U) \simeq \mathfrak{g}$ as the Lie superalgebras isomorphism, by an isomorphism mapping ϕ,*

$\phi:\ L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$,

$$D(T(-1,-1),T(-1,-1)) := \left\{ \begin{pmatrix} L(x,y) & -Id \\ Id & -L(y,x) \end{pmatrix} \right\}_{span} \longrightarrow$$

$$E_{-\rho} + ad([x,[y,E_\rho]]) + E_\rho,$$

$$T(-1,-1) := \left\{ \binom{x}{y} \middle| x, y \in U \right\}_{span} \longrightarrow x - [y, E_\rho].$$

where $< x, y > E_{-\rho} = -[x, y]$, $\forall x, y \in U$. *Note that* $< x, y > E_\rho = -[[x, E_\rho], [y, E_\rho]]$.

Proof. Indeed, it it is enough to show that

$$\phi(\left[\binom{a}{b}\binom{c}{d}\binom{e}{f}\right]) = (\left[\phi\binom{a}{b}\phi\binom{c}{d}\phi\binom{e}{f}\right]) = (\left[\left[\phi\binom{a}{b}, \phi\binom{c}{d}\right], \phi\binom{e}{f}\right]).$$

This is a straight fowrad calculations, but we omit it.

Appendix IV (the case of $\varepsilon = 1 and\ \delta = -1$**).** In this appendix IV, we describe another correspondence of $L(U)$ and $\mathfrak{g}$ as in appendix III.

For simple Lie superalgebras $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and the root system $\triangle$, then we have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \triangle} \mathfrak{g}_\alpha$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

And we intoroduce notations as follows;

the heighest root ρ ($\rho = 2\alpha_1 + \sum_{i \neq 1} n_i \alpha_i$), α_i is the simple roots,

$\triangle_0^* = \{\alpha \in \triangle | \alpha = \sum_{i \neq 1} m_i\ \alpha_i,\ m_i$: nonnegative integer$\}$,

$\triangle_1^* = \{\alpha \in \triangle | \alpha = \alpha_1 + \sum_{i \neq 1} k_i\ \alpha_i,\ k_i$: nonnegative integer$\}$,

$\triangle_2^* = \{\alpha \in \triangle | \alpha = 2\alpha_1 + \sum_{i \neq 1} l_i\ \alpha_i,\ l_i$: nonnegative integer$\}$.

Note that $\rho \in \triangle_2^*$, however, note that for $A(n-1,1) = sl(n|2)$ type's case with (n^2+4n+3) dimension, the highest root is $\rho = \alpha_1 + \alpha_n + \sum_{i \neq 1,n} m_i \alpha_i$, m_i is nonnegative integers.

Then based on these notations, we have a decomposition;

$\mathfrak{g} = \sum_{i=-2}^{2} \mathfrak{g}_i$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$,

where $\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \triangle_0^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 2} = \sum_{\alpha \in \triangle_2^*} \mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \triangle_1^*} \mathfrak{g}_{\pm\alpha}$.

We consider an endomorphism τ of $\mathfrak{g}$ satisfying $\tau^2 = id$ and $\tau([X,Y]) = [\tau(X), \tau(Y)], \forall X, Y \in \mathfrak{g}$, that is, $\tau : \mathfrak{g}_{-i} \to \mathfrak{g}_i$, by straightfoward calculatins, we obtain the following.

Theorem A.7. *The vector space* $U = \mathfrak{g}_{-1}$ *has a structure of (1,-1)-Freudenthal-Kantor triple system with respect to the triple product* $L(x,y)z := < xyz > = [[x, \tau(y)], z]$, $\forall x, y, z \in U$.

Hence we can verify the following theorem.

Theorem A.8. *For Lie superalgebras* $L(U)$ *and* $\mathfrak{g}$ *as in Theorem A.7, we have*

$L(U) \simeq \mathfrak{g}$ *as the Lie superalgebras isomorphism, by an isomorphism mapping* ϕ,

$$\phi:\ L(U) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

$$D(T(1,-1),T(1,-1)) := \left\{ \begin{pmatrix} L(x,y) & -K(a,b) \\ -K(c,d) & L(y,x) \end{pmatrix} \right\}_{span} \longrightarrow$$

$$[a,b] + ad([x,\tau(y)]) + [\tau(c),\tau(d)],$$

$$T(1,-1) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x,y \in U \right\}_{span} \longrightarrow x + \tau(y).$$

where $L(x,y)z =< xyz >$, $K(x,y)z =< xzy > + < yzx >$, $\forall x,yz \in U$.
Remark *Let* $(U, < xyz >)$ *be a* (ε,δ)*-FKTS. If* $J \in End\, U$ *satisfied* $J^2 = Id$ *and* $J < xyz >=< JxJyJz >$, *then* $(U,\{xyz\})$ *is a* $(-\varepsilon,\delta)$*-FKTS with respect to the product* $\{xyz\} =< xJyz >$.

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